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Alternate Approach for Aerospacecraft Design Sensitivities

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In Ref. 1, the classical indirect method of the calculus of variations is used to determine the optimum performance for a vehicle of given design. In Ref. 2, this work is extended by treating design variables as special control variables, and a fundamental variational relation of Leitmann[3] is used to deduce tradeoff relations or sensitivities between the performance index and the design variables. The purpose of this note is to offer an alternate approach to the derivation of these sensitivities.

Classical Variational Problem

Reference 1 considers the problem of determining the control variable histories

$$u_j(t) \qquad j = 1, 2, \ldots n \tag{1}$$

and the state variable histories

$$x_i(t) \qquad i = 1, 2, \ldots m \tag{2}$$

subject to the satisfaction of the differential equations

$$\dot{x}_i = f_i(t, x_i, u_j) \qquad i = 1, 2, \ldots m \qquad j = 1, 2, \ldots n \tag{3}$$

and the end conditions

$$\Psi_r[t^i, t^f, x_i(t^i), x_i(t^f)] = 0 \qquad r = 1, 2, \ldots q < 2m + 2 \tag{4}$$

in order to minimize a given performance index

$$G[t^i, t^f, x_i(t^i), x_i(t^f)] \tag{5}$$

The solution to this problem is secured by introducing m Lagrange multipliers

$$\lambda_1, \lambda_2 \ldots \lambda_m \tag{6}$$

and defining the generalized Hamiltonian by

$$H = \sum_{i=1}^{m} \lambda_i f_i \tag{7}$$

An optimum trajectory must satisfy the Euler-Lagrange equations

$$\left.\begin{aligned} \dot{x}_i &= \partial H/\partial \lambda_i \\ \dot{\lambda}_i &= \partial H/\partial x_i \end{aligned}\right\} \qquad (i = 1, 2, \ldots m) \tag{8, 9}$$

$$0 = \partial H/\partial u_j \qquad (\ = 1, 2, \ldots n) \tag{10}$$

and the transversality condition

$$\left[-H(t^s)dt^s + \sum_{i=1}^{m} \lambda_i(t^s)dx_i(t^s)\right]_{s=i}^{s=f} + dG = 0 \tag{11}$$

where (11) must hold for all sets of differentials

$$dt^i, dx_i(t^i), dt^f, dx_i(t^f) \qquad i = 1, 2, \ldots m \tag{12}$$

which satisfy

$$d\Psi_r = 0 \qquad r = 1, 2, \ldots q \tag{13}$$

The system composed of the constraining equations (4) and the Euler-Lagrange equations (8–10) is subject to $2m + 2$ boundary conditions for the optimum choice of the $2m + 2$ initial and final variables

$$t^i, t^f, x_i(t^i), x_i(t^f) \qquad i = 1, 2, \ldots m$$

Of these, q are supplied by (4) and the remaining $2m + 2 - q$ are supplied by the transversality conditions (11–13). The maximum number of restraints [Eq. (4)] that can be considered is $2m + 1$. In this latter case, one free variable is left to be determined from the transversality condition.

Fundamental Formula

If J denotes the minimum value of G subject to the given end conditions [Eq. (4)], then the formula of Leitmann[3]‡ is given by

$$dJ = dG + \left[-H^s dt^s + \sum_{i=1}^{m} \lambda_i(t^s)dx_i(t^s)\right]_{s=i}^{s=f} - \int_{t^i}^{t^f} \sum_{j=1}^{n} \frac{\partial H}{\partial u_j}\,\delta u_j\,dt \tag{14}$$

Along an optimum path, $dJ = 0$. The last term in (4) vanishes because of Eq. (10), and the first two terms combine with the given end conditions [Eq. (13)] to make $dJ = 0$.

Once an optimum path has been obtained, one may now go back and examine (14) to see what effect small perturbations on one or more of the state variables will have on J. To this end, Leitmann[3] assumes that the end values t^i, t^f, $x_\rho(t^i)$, $x\rho(t^f)$, $1 \leq \rho \leq m$, do not occur in G or the end conditions [Eq. (4)], and then calculates the effects of small perturbations

$$t^i + dt^i \qquad t^f + dt^f \qquad x_\rho(t^i) + dx_\rho(t^i) \qquad x_\rho(t^f) + dx_\rho(t^f)$$

on the performance index. In particular, because of (10) and (11), all terms of the right member of (14) vanish except those containing dt^i, dt^f, $dx_\rho(t^i)$, $dx_\rho(t^f)$. Thus (14) becomes

$$dJ = -H(t^f)dt^f + H(t^i)dt^i + \lambda_\rho(t^f)dx_\rho{}^f - \lambda_\rho(t^i)\,dx_\rho{}^i \tag{15}$$

and as a consequence one deduces that

$$\partial J/\partial t^f = -H(t^f) \qquad \partial J/\partial t^i = H(t^i) \tag{16}$$

$$\partial J/[\partial x_\rho(t^i)] = -\lambda_\rho(t^i) \qquad \partial J/[\partial x_\rho(t^f)] = \lambda_\rho(t^f) \tag{17}$$

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‡ For simplicity we exclude the case where the trajectory has corner points. Also, there are no equality nor inequality restraints on t, x_i, u_j.

Control Variable Approach

The procedure that Thelander uses to deduce tradeoff relations between J and design variables is to treat the design variables, say $b_1, b_2, \ldots b_p$, as special control variables. Thus, if we introduce p new control variables

$$u_{n+\alpha}(t) = b_\alpha \qquad \alpha = 1, 2, \ldots p \tag{18}$$

into the f_i's of Eq. (3) then we have a new variational problem in which p extra quantities [Eq. (18)] are to be determined. Since for most problems the b's will remain fixed at some constant value over the trajectory, one must now take several precautions. First one solves the variational problem with the b's determined at some nominal value. During this process, while from (10)

$$\partial H/\partial u_j = 0 \qquad j = 1, 2, \ldots n \tag{19}$$

one must remember that

$$\partial H/\partial u_{n+\alpha} \neq 0 \qquad \alpha = 1, 2, \ldots p \tag{20}$$

Thus, in (14), all terms cancel out in the right member except those due to (20). Consequently,

$$dJ = -\int_{t^i}^{t^f} \sum_{\alpha=1}^{p} \frac{\partial H}{\partial u_{n+\alpha}} \delta u_{n+\alpha}\, dt \tag{21}$$

and one can immediately deduce that

$$\frac{\partial J}{\partial u_{n+\alpha}} = -\int_{t^i}^{t^f} \frac{\partial H}{\partial u_{n+\alpha}}\, dt \qquad \alpha = 1, 2, \ldots p \tag{22}$$

One disadvantage with this approach is that care must be taken in distinguishing between the performance and design optimizations. During the performance optimization the primary control variables

$$u_j(t) \qquad j = 1, 2, \ldots n$$

are free of choice at each time, and, according to the maximum principle, indeed must be chosen so that H is a maximum at each time t of the interval $t^i \leq t \leq t^f$. For the secondary control variables, the maximum principle does not hold since the $u_{n+\alpha}$'s are not free variables along the trajectory. In fact, we must assume that

$$\delta u_{n+\alpha} = 0 \qquad \alpha = 1, 2, \ldots p \tag{23}$$

to insure that $dJ = 0$.

After the performance optimization has been completed, we restore (23) to their original meanings for the design optimization. The maximum principle simply asserts that the $u_{n+\alpha}$'s must be chosen so that J is a minimum. If it happens that the quantities [Eq. (20)] all vanish, then the simultaneous optimization of design and performance has been achieved. If not, the tradeoffs [Eq. (22)] indicate the direction in which the $u_{n+\alpha}$'s must be perturbed to obtain an optimum.

A second disadvantage is that very often design variables may occur in the performance criterion G or even in the end conditions. In such a case, special provisions would have to be made in order to attain correct results.

State Variable Approach

An alternate technique is to treat the design variables [Eq. (18)] as new state variables. In this approach, one introduces p new state variables

$$x_{m+\alpha}(t) = b_\alpha \qquad \alpha = 1, 2, \ldots p \tag{24}$$

with p new end conditions

$$\Psi_{q+\alpha} = x_{m+\alpha}(t^f) - x_{m+\alpha}(t^i) = 0 \qquad \alpha = 1, 2, \ldots p \tag{25}$$

and p new differential equations

$$\dot{x}_{m+\alpha} = 0 \qquad \alpha = 1, 2, \ldots p \tag{26}$$

When Eqs. (25) and (26) are appended to Eqs. (4) and (3), respectively, we have a classical problem of Mayer involving $m + p$ state variables and $q + p$ end conditions. The particular advantage of this method is that we now have a problem in which all the known results of the calculus of variations can be applied; no special procedures are required. This technique of treating parameters as state variables is due to Hestenes[4] and was later discussed by Cicala.[5]

For the problem at hand, the end conditions [Eq. (25)] now lead to p new transversality conditions

$$\lambda_{m+\alpha}(t^f) - \lambda_{m+\alpha}(t^i) = 0 \qquad \alpha = 1, 2, \ldots p \tag{27}$$

For some nonoptimum choice of the design variables [Eq. (24)], Eq. (14) becomes

$$dJ = \sum_{\alpha=1}^{p} [\lambda_{m+\alpha}(t^f) - \lambda_{m+\alpha}(t^i)] dx_{m+\alpha} \tag{28}$$

where

$$dx_{m+\alpha} = dx_{m+\alpha}(t^i) = dx_{m+\alpha}(t^f) \qquad \alpha = 1, 2, \ldots p \tag{29}$$

Since

$$\dot{\lambda}_{m+\alpha} = -\partial H/\partial x_{m+\alpha} \tag{30}$$

it follows that integration of (30) between t^i and t^f and substitution of (30) in (28) leads to the tradeoffs

$$\frac{\partial J}{\partial x_{m+\alpha}} = -\int_{t^i}^{t^f} \frac{\partial H}{\partial x_{m+\alpha}}\, dt \qquad \alpha = 1, 2, \ldots p \tag{31}$$

This is equivalent to Thelander's result (22).

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